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Rotating rotated branes

George Papadopoulos

DAMTP, Silver Street, University of Cambridge
Cambridge CB3 9EW
E-mail: g.papadopoulos@damtp.cam.ac.uk

Abstract: We present a class of spacetime rotations that preserve a proportion of spacetime supersymmetry. We then give the rules for superposing these rotations with various branes to construct rotating brane solutions which preserve exotic fractions of supersymmetry. We also investigate the superposition of rotations with intersecting branes at angles and we find new rotating intersecting branes at angles configurations. We demonstrate this with two examples of such solutions one involving intersecting NS-5-branes on a string at Sp(2) angles superposed with fundamental strings and pp-waves, and the other involving intersecting M-5-branes on a string at Sp(2) angles superposed with membranes and pp-waves. We find that the geometry of some of these solutions near the intersection region of every pair of 5-branes is $AdS_3 \times S^3 \times S^3 \times \mathbb{E}$ and $AdS_3 \times S^3 \times S^3 \times \mathbb{E}^2$, respectively. We also present a class of solutions that can be used for null string and M-theory compactifications preserving supersymmetry.

Keywords: M-Theory, p-branes, Black Holes in String Theory.

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Contents

1. Introduction

Many of the insights in the non-perturbative structure of the superstrings and in the understanding of M-theory have been found by investigated the soliton-like solutions of the ten- and eleven-dimensional supergravity theories [1, 2]. Most of the attention so far has been focused on supersymmetric solutions, i.e. those that they preserve a proportion of the supersymmetry of the underlying theory. It is remarkable that a large class of supersymmetric solutions of the supergravity theories can be constructed by superposing 'elementary' soliton solutions which preserve 1/2 of the spacetime supersymmetry [3, 4, 5]. These elementary solutions are the various brane solutions of supergravity theories, the pp-wave and KK-monopole. After such a superposition, the resulting solutions have the interpretation of intersecting branes or branes ending on other branes, and whenever appropriate, in the background of a pp-wave or a KK-monopole [6]. Recently it has been realized that angular momentum can be superposed to brane or to intersecting brane configurations in such way that a proportion of spacetime supersymmetry is preserved [7, 8, 9, 10]. The resulting configurations have the interpretation of rotating branes or rotating intersecting branes. Such backgrounds have also been consider in the context of sigma models as exact solutions of string theory [11].

In this paper, we shall present a systematic investigation of the type of rotations that can be added to branes and intersecting branes to preserve a fraction of spacetime supersymmetry. We shall find that in many cases supersymmetric rotations are associated with solutions of Maxwell field and Killing spinor equations on a possibly curved background. In turn, supersymmetric solutions of Maxwell field equations can be found by integrating certain BPS-like conditions in various dimensions which are associated to certain subalgebras of so(8). A list of such subalgebras includes spin(7), g_2, sp(2) and su(n) for $n = 2, 3, 4$. We shall give the explicit solutions of the Maxwell equations associated with all these subalgebras. Then we shall consider the associated spacetime solution. We shall find that such spacetimes are asymptotically flat with zero mass and momentum but they have non-vanishing angular momentum. Some of these spacetimes could be thought as 'elementary', for example those associated with the subalgebras spin(7) and su(n) for $n = 2, 3, 4$, and so they can be superposed with branes to give new solutions with the interpretation of rotating branes. However unlike the 'elementary' branes, pp-waves and KK-monopoles, spacetimes with only 'elementary' rotation may preserve less than 1/2 of spacetime supersymmetry.[1] Some other rotations can be constructed by a non-parallel superposition of 'elementary' ones. Such rotating spacetimes are associated with sp(2) and g_2 subgalgebras. We shall find that these composite rotations can be superposed to intersecting branes at angles. Thus we construct solutions with the interpretation of rotating rotated intersecting branes. In what follows, we shall refer to intersections or superpositions of branes at G-angles as G-intersections or G-superpositions, respectively. We shall present two main examples of such rotating rotated intersecting branes. These examples involve the Sp(2)-intersecting NS-5-branes and the Sp(2)-intersecting M-5-branes on a string of [12, 13], respectively. In addition, pp-waves will be added to both configurations as well as strings in the former and membranes in the latter. Rotations will also be investigated in the background of Sp(2)-superpositions

[1] This indicates that there may be another interpretation for this rotations probably in terms of more 'elementary' objects. However we shall not pursue this further here.

of KK-monopoles which is described by a toric eight-dimensional hyper-Kähler metric. The rotation in this case is related to the two tri-holomorphic vector fields of this geometry. Rotations will be examined in the background of Ricci-flat cones with special holonomy and their various superpositions with strings, membranes and pp-waves will be also considered. In particular, this will add rotation to the solutions found in [14, 15]. The proportion of spacetime supersymmetry preserved by all the above configurations will also be given. We shall find that in many cases the solutions preserve various exotic fractions of supersymmetry.

The geometries near the intersection region (near horizon) of the above rotating Sp(2)-intersecting 5-brane configurations will be investigated. We shall find that the geometry near the intersection region of every pair of NS-5-branes involved in a Sp(2)-intersection on a string is $\mathbb{E}^{(1,3)} \times S^3 \times S^3$. This geometry is the same as that for the associated orthogonal intersection [16, 17]. Then we shall find that the geometry at the intersection region of rotating Sp(2)-intersecting NS-5-branes superposed with pp-waves and strings is $AdS_3 \times S^3 \times S^3 \times \mathbb{E}$. In addition, the geometry at the intersection region of rotating Sp(2)-intersecting M-5-branes superposed with pp-waves and membranes is $AdS_3 \times S^3 \times S^3 \times \mathbb{E}^2$. To show this in the above two cases, we shall present approximate solutions which have the desirable behaviour near the intersection regions. We remark that these near horizon geometries are the same as those found in [16, 17, 10] for the associated orthogonal intersections. Finally we shall show that spaces with topology $\mathbb{R}^n \times \mathcal{L}$ solve the field equations of D=11 and D=10 supergravities, where $\mathcal{L}$ is a principal U(1) bundle over either tori, or Calabi-Yau or other manifolds with special holonomy. We shall argue that such spaces can be used to compactify strings and M-theory along directions for which one is null. This adapts the null compactifications investigated for example in [18] to strings and M-theory.

This paper has been organized as follows: In section 2, we present the ansatz for our solutions in the context of type II strings. In section 3, we investigate spacetimes with pure rotation and explain the relation between rotations and subalgebras of so(8). We then present new solutions with the interpretation of rotating strings and rotating NS-5-branes. Superpositions of pp-waves are also considered. In section 4, we investigate rotations in the background of toric hyper-Kähler manifolds. In section 5, we construct new solutions with the interpretation of rotating rotated NS-5-branes and then superpose them with strings and waves. In section 6, we investigate rotations in the background of Ricci-flat cones with special holonomy and then superpose them with strings and membranes. In section 7, we give an M-theory interpretation to our solutions by lifting them to eleven dimensions. In section 8, we find the geometry of the rotating rotated 5-brane solutions near the intersection of every pair of 5-branes involved in the configuration. In section 9, we find the compact solutions of strings and M-theory that can be used for null compactifications. In section 10, we give our conclusions and in appendix A we investigate the type II T-duality and IIB S-duality properties of our solutions.

2. Common sector rotations

The common sector fields or the NS⊗NS fields of type II and heterotic string theories include the spacetime metric g, a three-form field strength H and the dilaton ϕ. The field equations of the associated effective supergravity theories can be consistently truncated to this sector and they are as follows:

$$\begin{aligned} R_{MN} - H_{MPQ}H_N{}^{PQ} + 2\nabla_M\partial_N\phi &= 0 \\ \nabla_P\left(e^{-2\phi}H^{PMN}\right) &= 0\,, \end{aligned} \tag{2.1}$$

where we have set all fermions to zero and R_{MN} is the Ricci tensor of the metric g_{MN}; $M, N, P, Q = 0, \ldots, 9$. There is another field equation that of the dilaton but it is implied by the above two equations (up to a constant).

The Killing spinor equations for type II theories that involve only the above fields are

$$\begin{aligned} \nabla^{(\pm)}_M\epsilon_\pm &= 0 \\ \left(\Gamma^M\partial_M\phi \mp \frac{1}{3!}H_{MNP}\Gamma^{MNP}\right)\epsilon_\pm &= 0\,, \end{aligned}$$

where $\epsilon_\pm$ are 16-component Killing spinors and the connections of the covariant derivatives $\nabla^{(\pm)}$ with torsion are

$$\Gamma^{(\pm)M}{}_{NP} = \Gamma^M{}_{NP} \pm H^M{}_{NP}\,;$$

$\Gamma^M{}_{NP}$ is the Levi-Civita connection. In IIA and IIB supergravities, ϵ_+ and ϵ_- have either the opposite or the same chirality, respectively. For the heterotic string, we simply truncate the above four Killing spinor equations to two which are those of either ϵ_+ or ϵ_-.

We shall seek solutions that describe the superposition of rotations with configurations of the common sector. These configurations include the fundamental string, the NS-5-brane, the pp-wave and the ten-dimensional KK-monopole as well as their intersections or superpositions. To describe the ansatz of such configuration, we begin with a static solution on $\mathcal{M}_{(10)} = \mathbb{E}^{(1,1)} \times \mathcal{M}_{(8)}$ of the NS⊗NS sector of the form

$$\begin{aligned} ds^2 &= ds^2(\mathbb{E}^{(1,1)}) + ds^2_{(8)} \\ H &= H_{(8)} \\ e^{2\phi} &= e^{2\phi_{(8)}}\,, \end{aligned}$$

where

$$\begin{aligned} ds^2_{(8)} &= \gamma_{ab}dx^a dx^b \\ H_{(8)} &= \frac{1}{3!}h_{abc}dx^a\wedge dx^b\wedge dx^c \end{aligned}$$

and $e^{2\phi_{(8)}}$ depend only on the coordinates $\{x^a; a = 1, \dots, 8\}$ of the eight-dimensional manifold $\mathcal{M}_{(8)}$. Such solution describes configurations that involve the NS-5-brane, the KK-monopole and their various intersections and superpositions. It remains to add a fundamental string, a pp-wave and a rotation into this background. For this we introduce a one-form A, and two functions g_1 and g_2 on $\mathcal{M}_{(8)}$ which will be associated with the rotation, the fundamental string and pp-wave, respectively. Then we write the ansatz

$$\begin{aligned} ds^2 &= 2g_1^{-1}dv(-du + A + g_2 dv) + ds^2_{(8)} \\ H &= \lambda dv \wedge d(g_1^{-1}A) + \lambda du \wedge dv \wedge dg_1^{-1} + H_{(8)} \\ e^{2\phi} &= g_1^{-1}e^{2\phi_{(8)}}\,, \end{aligned} \tag{2.2}$$

where λ is a real number and u, v are light-cone coordinates. The functions g_1, g_2, the one-form A and the constant λ will be determined by solving the Killing spinor and field equations of supergravity. The metric ds^2 and the three-form field strength H are invariant under the gauge transformations

$$\begin{aligned} A &\to A + d\alpha \\ u &\to u + \alpha\,, \end{aligned}$$

where $\alpha = \alpha(x)$ is the parameter of the gauge transformations. Our proposed solutions are therefore associated with a principal U(1) bundle $\mathcal{L}$ over $\mathcal{M}_{(8)}$ with fibre coordinate u. The presence of the functions g_1 and g_2 in (2.2) to describe a fundamental string and a pp-wave is standard and we shall not explain it further. The addition of the gauge potential A to describe rotation is motivated from the way that angular momentum is computed at infinity. Moreover A is taken to be independent from the worldvolume coordinates of the associated brane since angular momentum in the context of branes in measured at the transverse spatial infinity; this is a direct analogue to the case of black holes where angular momentum is measured at the spatial infinity. The expression of the angular momentum of a p-brane is given in the section below.

To find solutions to Killing spinor equations, it is convenient to introduce the frame

$$\begin{aligned} e^{\underline{v}} &= dv \\ e^{\underline{u}} &= g_1^{-1}(-du + A + g_2 dv) \\ e^{\underline{a}} &= \tilde{e}^{\underline{a}}\,, \end{aligned}$$

where $\tilde{e}$ is a frame of $ds^2_{(8)}$. The metric and 3-form field strength in terms of this frame are written as

$$\begin{aligned} ds^2 &= 2e^{\underline{v}}e^{\underline{u}} + \delta_{\underline{ab}}e^{\underline{a}}e^{\underline{b}} \\ H &= \lambda e^{\underline{v}} \wedge de^{\underline{u}} + H_{(8)}\,. \end{aligned}$$

A direct computation reveals that the Killing spinor equations reduce to

$$\begin{aligned}
\left(-\frac{1}{2}\pm\lambda\right) g_1^{-1}\partial_b g_1 \tilde{e}^j{}_{\underline{a}}\Gamma^{\underline{va}}\epsilon_\pm &= 0 \\
\left(\frac{1}{2}\pm\lambda\right) g_1^{-1}F_{ch}\tilde{e}^c{}_{\underline{a}}\tilde{e}^h{}_{\underline{b}}\Gamma^{\underline{ab}}\epsilon_\pm &= 0 \\
g_1^{-1}\partial_b g_2 \tilde{e}^b{}_{\underline{a}}\Gamma^{\underline{va}}\epsilon_\pm &= 0 \\
\left(\frac{1}{2}\pm\lambda\right) g_1^{-1}\partial_b g_1 \tilde{e}^b{}_{\underline{a}}\Gamma^{\underline{ua}}\epsilon_\pm &= 0 \\
D^{(\pm)}_a\epsilon_\pm+\frac{1}{2}\left(\frac{1}{2}\pm\lambda\right) g_1^{-1}F_{ac}\tilde{e}^c{}_{\underline{b}}\Gamma^{\underline{vb}}\epsilon_\pm+& \\
+\frac{1}{4}\left(\frac{1}{2}\mp\lambda\right) g_1^{-1}\partial_a g_1(\Gamma^{\underline{vu}}-\Gamma^{\underline{uv}})\epsilon_\pm &= 0 \\
\left(\Gamma^{\underline{a}}\tilde{e}^a{}_{\underline{a}}\partial_a\phi_{(8)}\mp\frac{1}{3!}h_{abc}\tilde{e}^a{}_{\underline{a}}\tilde{e}^b{}_{\underline{b}}\tilde{e}^c{}_{\underline{c}}\Gamma^{\underline{abc}}\right)\epsilon_\pm-& \\
-\frac{1}{2}\Gamma^{\underline{a}}\tilde{e}^a{}_{\underline{a}}g_1^{-1}\partial_a g_1\left(1\mp\lambda[\Gamma^{\underline{uv}}-\Gamma^{\underline{vu}}]\right)\epsilon_\pm\mp& \\
\mp\frac{1}{2}\lambda\, g_1^{-1}F_{ab}\tilde{e}^a{}_{\underline{a}}\tilde{e}^b{}_{\underline{b}}\Gamma^{\underline{v}}\Gamma^{\underline{ab}}\epsilon_\pm &= 0\,,
\end{aligned} \tag{2.3}$$

where $D^{(\pm)}_a$ are the covariant derivatives with torsion of $\mathcal{M}_{(8)}$ and

$$F_{ab}=2\partial_{[a}A_{b]}\,.$$

We have also taken the Killing spinors $\epsilon_\pm$ to be independent from the light-cone coordinates u, v.

To determine g_1, g_2 and A, we have also to use the field equations (2.1). These are dramatically simplified provided that we choose $\lambda=\pm1/2$. As we shall see later for these values of λ and for certain choices of $\mathcal{M}_{(8)}$, our solutions will preserve a proportion of spacetime supersymmetry. Substituting our ansatz into the field equations for

$$\lambda=-\frac{1}{2}\,,$$

we find

$$\begin{aligned}
\partial_a\left(\sqrt{\gamma}\,e^{-2\phi_{(8)}}\gamma^{ab}\partial_b g_1\right) &= 0 \\
\partial_a\left(\sqrt{\gamma}\,e^{-2\phi_{(8)}}\gamma^{ab}\partial_b g_2\right) &= 0 \\
\partial_b\left(\sqrt{\gamma}\,e^{-2\phi_{(8)}}F^{ba}\right)-\sqrt{\gamma}\,e^{-2\phi_{(8)}}h^{bca}F_{bc} &= 0\,,
\end{aligned} \tag{2.4}$$

where indices are raised and lowered using the metric $ds^2_{(8)}$ on $\mathcal{M}_{(8)}$ and γ is the determinant of $ds^2_{(8)}$. Note that if we choose $\lambda=+1/2$, then the relative sign between the two terms in the last field equation above changes. The two choices for λ are symmetric, so without loss of generality, we shall set $\lambda=-1/2$ for the rest of the paper. In what follows we shall explore the solutions of the Killing spinor and field equations for various choices of $\mathcal{M}_{(8)}$ and investigate their applications in strings and M-theory.

2.1 Asymptotics and angular momentum

Ten-dimensional spacetimes with the interpretation of p-branes are asymptotically $\mathbb{E}^{(1,9)} = \mathbb{E}^{(1,p)} \times \mathbb{E}^{9-p}$ at the transverse spatial infinity, where $\mathbb{E}^{(1,p)}$ are the world-volume directions and $\mathbb{E}^{9-p}$ are the transverse directions of the brane. The mass per unit volume (or tension) M and the charge Q per unit volume of a p-brane are given as integrals over the $S^{8-p} \subset \mathbb{E}^{9-p}$ sphere at the transverse spatial infinity. Writing the Einstein frame metric as

$$g_{MN} = \eta_{MN} + h_{MN}\,,$$

where η is the metric of ten-dimensional Minkowski spacetime, the angular momentum of a p-brane per unit volume is given by

$$\mathcal{J}_{ab} = K_p \lim_{r\to\infty} \int_{S^{8-p}} \Big[-x_a\partial_c h_{0b} + x_a\partial_t h_{cb} + h_{0b}\delta_{ac} - (a,b)\Big] n^c r^{8-p} d\Omega\,,$$

where $\{x^a; a = 1,\dots,9-p\}$ are the transverse coordinates of the brane, $r = \sqrt{\delta_{ab}x^a x^b}$, $n^a = x^a/r$ is the outward normal vector of the S^{8-p} sphere at infinity, K_p is a constant and $d\Omega$ is the volume form of S^{8-p}.

Our ansatz (2.2) is invariant under the action of the Killing vector field $X = \partial_t$. Since we are considering backgrounds that involve either a fundamental string or a NS-5-brane with possibly a wave and a rotation, $p = 1$ or $p = 5$, respectively. In the case of fundamental string, we take $M_{(8)} = \mathbb{E}^8$ and require that at the transverse spatial infinity

$$\begin{aligned} g_1 &= 1 + O\left(\frac{1}{r^6}\right) \\ g_2 &= 0 + O\left(\frac{1}{r^6}\right) \\ A &= 0 + O\left(\frac{1}{r^7}\right) \end{aligned}$$

and similarly for the 5-brane. In particular for the 5-brane

$$A = 0 + O\left(\frac{1}{r^3}\right)\,.$$

The angular momentum in both cases can be written as

$$\mathcal{J}_{ab} = K_p \lim_{r\to\infty} \int_{S^{8-p}} \Big[-x_a\partial_c A_b + A_b\delta_{ac} - (a,b)\Big] n^c r^{8-p} d\Omega\,. \qquad (2.5)$$

A convenient choice for K_p is $K_p = -1/((7-p)V(S^{8-p}))$, where $V(S^{8-p})$ is the volume of a (8-p)-dimensional sphere of radius one. For solutions that describe intersecting branes with rotation, the behaviour at infinity is rather complicated. However we can compute the rotation of each brane involved in the intersection by taking the rest of the branes at infinity. In this case the expression for the angular momentum is again given by (2.5).

3. Pure rotation

The geometric conditions imposed by the requirement of supersymmetry and the field equations on the connection A can be most conveniently illustrated by taking $\mathcal{M}_{(8)} = \mathbb{E}^8$, $g_1 = 1$ and $g_2 = 0$. In this case, our ansatz (2.2) reduces to

$$
\begin{aligned}
ds^2 &= 2dv(-du + A) + ds^2(\mathbb{E}^8) \\
H &= -\frac{1}{2} dv \wedge dA\,;
\end{aligned}
$$

($\lambda = -1/2$). The angular momentum of the associated spacetime is induced by A.

A direct substitution into Killing spinor equations (2.3) reveals that there are solutions provided that

$$
\begin{aligned}
\Gamma^{\underline{v}} \epsilon_+ &= 0 \\
F_{ab}\Gamma^{ab}\epsilon_- &= 0\,,
\end{aligned}
\tag{3.1}
$$

and $\epsilon_\pm$ constant; we have made no distinction between coordinate and frame indices on $\mathbb{E}^8$. Moreover the field equations reduce to

$$
\partial^a F_{ab} = 0\,.
$$

Therefore F satisfies the Killing spinor and field equations of Maxwell theory on $\mathbb{E}^8$.

The first Killing spinor equation can be easily solved giving eight Killing spinors. To find the solutions of the second Killing spinor equation, we first observe that the space of two forms on $\mathbb{E}^n$ can be identified with the Lie algebra of $\mathrm{SO}(n)$, i.e.

$$
\bigwedge^2(\mathbb{E}^n) = \mathrm{so}(n)\,.
$$

Then we view the Maxwell Killing spinor equation as the equation for a spinor singlet under an infinitesimal orthogonal rotation with parameter F. For generic orthogonal rotations in $\mathbb{E}^8$, there are no such singlets and so the solution does not admit any additional Killing spinors. So supergravity solutions associated with a generic solution of the Maxwell field equations of A preserve $1/4$ of spacetime supersymmetry. However more supersymmetry can be preserved by such supergravity solutions, if F takes values in an appropriate subalgebra h of $\mathrm{so}(8)$. Such subalgebras are those that are associated with infinitesimal orthogonal rotations which leave certain spinors invariant (see also [19, 20]). Examples of such subalgebras of $\mathrm{so}(8)$ are $h = \mathrm{su}(2), \mathrm{su}(3), \mathrm{su}(4), \mathrm{sp}(2), \mathrm{g}_2$ and $\mathrm{spin}(7)$. In the table below we have summarized the various fractions of supersymmetry preserved by the solution for every subalgebra of $\mathrm{so}(8)$.

For the computation of these fractions, we have first decomposed the sixteen-dimensional representations $\mathbf{16}_s$ and $\mathbf{16}_c$ of $\mathrm{Cliff}(\mathbb{E}^{(1,9)})$ under $\mathrm{spin}(1,1) \times \mathrm{Spin}(8)$ as

$$
\mathbf{16}_s \to (\mathbf{1}_s, \mathbf{8}_s) \oplus (\mathbf{1}_c, \mathbf{8}_c) \tag{3.2}
$$

$$
\mathbf{16}_c \to (\mathbf{1}_s, \mathbf{8}_c) \oplus (\mathbf{1}_c, \mathbf{8}_s)\,. \tag{3.3}
$$

Algebra	so(8)	spin(7)	g_2	sp(2)	su(4)	su(3)	su(2)
Susy	$\frac{1}{4}$	$\frac{9}{32}$	$\frac{10}{32}$	$\frac{11}{32}$	$\frac{10}{32}$	$\frac{3}{8}$	$\frac{1}{2}$

The solutions of the first Killing spinor equation in (3.1) are either in the subspace $(\mathbf{1}_s, \mathbf{8}_s)$ or in the subspace $(\mathbf{1}_s, \mathbf{8}_c)$ of $\mathbf{16}_s$ depending on whether $\Gamma^{\underline{v}}\epsilon_+ = 0$ is related to a anti-chiral or to a chiral projection of Cliff$(1,1)$, respectively. Without loss of generality we can choose the former case, so the Killing spinors are in $(\mathbf{1}_s, \mathbf{8}_s)$ subspace of $\mathbf{16}_s$. The remaining Killing spinors arise as singlets in the decomposition of $\mathbf{8}_s$ and $\mathbf{8}_c$ that appears in the second equation of (3.3) under the associated subalgebra h of so(8) in the table above. Similar computations have been done elsewhere (see for example [12]) and we shall not give more details here.

3.1 $h = \mathrm{su}(2), \mathrm{su}(3)$ and $\mathrm{su}(4)$

We begin the construction of explicit supergravity solutions by first taking F to take values in $h = \mathrm{su}(n)$ for $n = 2, 3, 4$. Such connections have support in a subspace $\mathbb{E}^{2n}$ of $\mathbb{E}^8$ and they are characterized by a complex structure K on $\mathbb{E}^{2n}$ which determines the embedding of SU(n) in SO$(2n) \subset$ SO(8). For $h = \mathrm{su}(2)$, A is either self-dual or anti-self-dual connection depending on the choice of complex structure K relative to the choice of orientation of $\mathbb{E}^4$. For $h = \mathrm{su}(n)$, $n = 3, 4$, A is a hermitean-Einstein connection on $\mathbb{E}^6$ and $\mathbb{E}^8$, respectively [22]. To continue, we choose complex coordinates $\{z^\alpha; \alpha = 1, \ldots, n\}$ with respect to K. The conditions on F for $h = \mathrm{su}(n)$ are

$$\begin{aligned} F_{\alpha\beta} &= 0 \\ \delta^{\alpha\bar\beta} F_{\alpha\bar\beta} &= 0\,. \end{aligned} \tag{3.4}$$

Note that the flat metric is hermitean with respect to K and that the above conditions on F imply the Maxwell field equations using the Jacobi identity.

To find the solutions of the above equations, we first observe that the first one implies that

$$\begin{aligned} A_\alpha &= \partial_\alpha U \\ A_{\bar\alpha} &= \partial_{\bar\alpha} \bar U\,, \end{aligned}$$

where U is a complex function. Substituting this into the second equation in (3.4), we find that

$$\delta^{\alpha\bar\beta}\partial_\alpha\partial_{\bar\beta}(U - \bar U) = 0\,.$$

This equation can be easily solved to find that

$$U - \bar U = if\,,$$

where

$$f = 1 + \sum_i^N \frac{\mu_i}{|z - z_i|^{2n-2}} \tag{3.5}$$

is a real harmonic function on $\mathbb{R}^{2n}$. The above equations do not determine the real part of U but this is expected since it is a gauge degree of freedom. So up to a gauge transformation, the solution is

$$\begin{aligned} A_\alpha &= i\partial_\alpha f \\ A_{\bar{\alpha}} &= -i\partial_{\bar{\alpha}} f \,, \end{aligned}$$

which can be rewritten in real coordinates as

$$A_a = K^b{}_a \partial_b f \,.$$

In four dimensions, the solution in this form has already been given in [23, 24]. Moreover it turns out that if the Kähler form of K is self-dual, then A is anti-self-dual and vice-versa. In fact in this case the solution can be generalized. For this we use that $h = \mathrm{su}(2) = \mathrm{sp}(1)$ and observe the hermitean-Einstein connection on F can be imposed with any of the three complex structures, or a linear combination of them, that define the embedding of sp(1) in so(4). We remark that independently from the choice of complex structure, F is either a self-dual or anti-self-dual two-form. Let $\{I_r; r = 1, 2, 3\}$ be the quaternionic structure that defines the embedding of sp(1) in so(4). Using the linearity of the equations and the remark above, the most general solution can be written as

$$A_a = \sum_i^N \sum_{r=1}^3 (I_r)^b{}_a \partial_b \frac{\mu_i^r}{|x - x_i|^2} \,.$$

We shall see that for the investigation the asymptotic behaviour of supergravity solutions, it is sufficient to consider spherically symmetric rotations. The rest of the solutions can be thought as 'parallel' superpositions of the spherically symmetric ones. For spherically symmetric solutions, we have

$$A_a = K^b{}_a \partial_b \frac{\mu}{|x|^{2n-2}} \,,$$

for $n = 3$ and $n = 4$, and

$$A_a = \sum_{r=1}^3 (I_r)^b{}_a \partial_b \frac{\mu^r}{|x|^2} \,,$$

for $n = 2$, respectively. The solutions for $n = 2$ and $n = 4$ can be used to add rotation to a NS-5-brane and to a fundamental string, respectively. The case $n = 3$ can be used to add rotation to a D-3-brane or a compactified fundamental string. The latter case will lead a rotating black hole in seven dimensions. Using the expression for the angular momentum (2.5), we find that

$$\mathcal{J}_{ab} = \mu K_{ab} \,,$$

for $n = 3, 4$, and

$$\mathcal{J}_{ab} = \sum_{r=1}^{3} \mu^r (I_r)_{ab} ,$$

for $n = 2$. In both cases, the angular momentum has one independent eigenvalue.

3.2 $h = \mathrm{sp}(2)$

Let $\{\mathbf{K}_r; r = 1, 2, 3\}$ be a quaternionic structure in $\mathbb{E}^8$ that determines the embedding of $\mathrm{Sp}(2)$ in $\mathrm{SO}(8)$. The condition that F is in $\mathrm{sp}(2)$ is equivalent to the condition that F is a (1,1) two-form with respect all three complex structures $\{\mathbf{K}_r; r = 1, 2, 3\}$, i.e.

$$F_{ab}(\mathbf{K}_r)^a{}_c(\mathbf{K}_r)^b{}_d = F_{cd}$$

(no summation over r). To find the conditions on A, it is convenient to introduce a coordinates $\{x^{i\mu}; i = 1, 2; \mu = 1, \ldots, 4\}$ on $\mathbb{E}^8$ such that

$$\mathbf{K}_r{}^{i\mu}{}_{j\nu} = \delta^i{}_j(K_r)^\mu{}_\nu ,$$

where $\{K_r; r = 1, 2, 3\}$ is a quaternionic structure is $\mathbb{E}^4$. Then since F is (1,1)-form with respect to all three complex structures $\{\mathbf{K}_r; r = 1, 2, 3\}$, we choose the first one and can write

$$A_{i\mu} = (K_1)^\nu{}_\mu \partial_{i\nu} f , \tag{3.6}$$

where H is a real function, as in the case of hermitean-Einstein connections in the previous section. This leads to an F which is (1,1)-form with respect to $\mathbf{K}_1$. It is straightforward to show that for F to be (1,1)-form with respect to the remaining complex structures [23] then

$$\begin{aligned} \delta^{\mu\nu}\partial_{i\mu}\partial_{j\nu} f &= 0 \\ \partial_{i\mu}\partial_{j\nu} f &= \partial_{j\mu}\partial_{i\nu} f \end{aligned}$$

Solving these equations,[2] we find that

$$A_{i\mu} = \sum_{\{(p,a)\}} \sum_{r=1}^{3} (K_r)^\nu{}_\mu \partial_{i\nu} \frac{\mu^r((p,a))}{|p_i x^{i\mu} - a^\mu|^2} , \tag{3.7}$$

where the parameters $\{p_i, i = 1, 2; a^\mu, \mu = 1, \ldots, 4\}$ of the solution are the real numbers. To write this solution, we have also used the fact that A can be written in three different but equivalent ways as (3.6) with respect to each complex structure $\{K_r; r = 1, 2, 3\}$.

The interpretation of the solutions (3.7) is as a superposition of four-dimensional abelian instantons (or anti-instantons) at $\mathrm{Sp}(2)$ angles. This follows from the analysis in [25] of the non-abelian case. Therefore the abelian gauge potentials (3.7) can be used to rotate intersecting brane configurations. This is achieved by using each

[2] For the non-abelian case see [21].

four-dimensional abelian instanton involved in the superposition to rotate a brane involved in the intersection. We shall verify this when we investigate the rotation of intersecting 5-branes at angles. If we take all the abelian instantons at infinity apart from one and compute the angular momentum of the associated spacetime, we shall find that

$$(\mathcal{J})_{ab} = \sum_{r=1}^{3} \mu^r (K_r)_{\mu\nu} ,$$

which is that of the $n = 2$ case of the previous section.

We finally remark that the condition that F is in sp(2) can also be written as self-duality like condition

$$\frac{1}{2}\Omega_{ab}{}^{cd} F_{cd} = F_{ab} ,$$

where

$$\Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r ,$$

where ω_r is the Kähler form of $\mathbf{K}_r$.

3.3 $h = \mathrm{spin}(7)$ and g_2

We shall first give the solutions for which F is in spin(7). For this, let φ be the 3-form with components the structure constants of imaginary unit octonions and ϕ be its dual in $\mathbb{E}^7$, i.e.

$$\phi_{ijk\ell} = \frac{1}{6}\epsilon_{ijk\ell}{}^{pqs}\varphi_{pqs} ,$$

where $i, j, k, \ell, p, q, s = 1, \ldots, 7$. Then a spin(7)-invariant self-dual 4-form ω in $\mathbb{E}^8$ can be defined as

$$\begin{aligned} \omega_{ijk8} &= \varphi_{ijk} \\ \omega_{ijk\ell} &= \phi_{ijk\ell} . \end{aligned}$$

For later use, we remark that the four-form ω satisfies

$$\omega^{abch}\omega_{defh} = 6\delta^{abc}_{def} - 9\delta^{[a}_{[d}\omega^{bc]}{}_{ef]} ,$$

where $a, b, c, d, e, f, h = 1, \ldots, 8$. We introduce the orthogonal projectors

$$\begin{aligned} (P_1)^{ab}{}_{cd} &= \frac{3}{4}\left(\delta^{ab}{}_{cd} + \frac{1}{6}\omega^{ab}{}_{cd}\right) \\ (P_2)^{ab}{}_{cd} &= \frac{1}{4}\left(\delta^{ab}{}_{cd} - \frac{1}{2}\omega^{ab}{}_{cd}\right) \end{aligned}$$

on $\bigwedge^2(\mathbb{E}^8)$. These projectors are associated with an orthogonal decomposition of $\bigwedge^2(\mathbb{E}^8)$ into a 21-dimensional subspace [26], which can be identified with spin(7), and a seven-dimensional subspace, respectively. The condition that F is in spin(7) is therefore $P_2 F = 0$ or equivalently

$$\frac{1}{2}\omega_{ab}{}^{cd}F_{cd} = F_{ab}\,.$$

To find solutions to this equation, we introduce the ansatz

$$A_a = v^i (I_i)^b{}_a \partial_b f\,,$$

where

$$\begin{aligned}(I_i)^8{}_j &= \delta_{ij}\\ (I_i)^j{}_8 &= -\delta_i^j\\ (I_i)^k{}_j &= \varphi_i{}^k{}_j\end{aligned}$$

and v is a (constant) vector in $\mathbb{E}^7$. The matrices $\{I_i; i=1,\dots,7\}$ satisfy $I_iI_j + I_jI_i = -2\delta_{ij}$ and so they give an irreducible representation of $\mathrm{Cliff}(\mathbb{R}^7)$ equipped with the negative definite inner product. Using the identities

$$\begin{aligned}\phi^{mijk}\varphi_{mpq} &= -6\delta^{[i}{}_{[p}\varphi^{jk]}{}_{q]}\\ \varphi^{mij}\varphi_{mk\ell} &= 2\delta^{ij}{}_{k\ell} - \phi^{ij}{}_{k\ell}\,,\end{aligned}$$

a direct substitution of the ansatz into the condition $P_2F = 0$ reveals that

$$\delta^{ab}\partial_a\partial_b f = 0\,,$$

for any choice of v. So f is a harmonic function in $\mathbb{E}^8$. Therefore for this class of solutions

$$A_a = \sum_{n=1}^{N}\sum_{i=1}^{7}(I_i)^b{}_a\partial_b\frac{\mu_n^i}{|x-x_n|^6}\,.$$

It is clear that the above abelian U(1) gauge fields can be used to rotate fundamental strings and M-2-branes. The spherically symmetric solution is

$$A_a = \sum_{i=1}^{7}(I_i)^b{}_a\partial_b\frac{\mu^i}{|x|^6}\,.$$

The angular momentum of the spacetime associated with this solution is

$$(\mathcal{J})_{ab} = \sum_{i=1}^{7}\mu^i (I_i)_{ab}\,.$$

Next we shall examine the case where F is in g_2. We again introduce the projectors

$$\begin{aligned}(P_1)^{ij}_{k\ell} &= \frac{2}{3}\left(\delta^{ij}{}_{k\ell} + \frac{1}{4}\phi^{ij}{}_{k\ell}\right)\\ (P_2)^{ij}_{k\ell} &= \frac{1}{3}\left(\delta^{ij}{}_{k\ell} - \frac{1}{2}\phi^{ij}{}_{k\ell}\right),\end{aligned}$$

which introduce an orthogonal decomposition of $\bigwedge^2(\mathbb{R}^7)$ into a 14-dimensional subspace, that can be identified with g_2, and a seven-dimensional subspace. Since P_2 projects onto the seven-dimensional subspace, the condition that F is in g_2 is $P_2(F) = 0$ or equivalently

$$\frac{1}{2}\omega_{ij}{}^{k\ell}F_{k\ell} = F_{ij}\,. \tag{3.8}$$

To solve this equation, we write the ansatz

$$A_i = v^k \varphi_k{}^j{}_i \partial_j f$$

in analogy with the spin(7) case above. Then for F to be in g_2, we find that

$$\varphi_{ij}{}^k v^\ell \partial_k \partial_\ell f - \varphi_{ijk} v^k \delta^{m\ell} \partial_m \partial_\ell f = 0\,.$$

Since the above equation is linear v, we can take with loss of generality v to have length one. To solve the above equation let us assume that $v = (0, \dots, 0, 1)$. Now if we take f to be independent[3] of x^7, then the above equation implies that f is harmonic on the hyperplane in $\mathbb{E}^7$ orthogonal to v. So for this choice of v a typical solution is

$$f = 1 + \sum_{n=1}^{N} \frac{\mu_n}{|\tilde{x} - \tilde{x}_n|^4}\,,$$

where $\tilde{x}$ are the first six coordinates of $\mathbb{E}^7$. For the most general solution of this type using the linearity of the condition (3.8) in A, we can sum over different choices of v. This leads to the following solution for A:

$$A_i = \sum_{\{v\}} \sum_{n=1}^{N} v^k \varphi_k{}^j{}_i \partial_j \frac{\mu_n}{|x_v - (x_v)_n|_v^4}\,, \tag{3.9}$$

where x_v are the coordinates of the hyperplane P_v orthogonal to v and $|\cdot|_v$ is the norm on P_v induced by the standard norm on $\mathbb{E}^7$.

The solutions of the condition that F is in g_2 has many similarities with the solutions we have found for the condition that F is in sp(2) in section (3.2). In the latter case, the expression for A is given by summing over four-dimensional subspaces of $\mathbb{E}^8$ while in the former the expression for A is given by summing over hyperplanes in $\mathbb{E}^7$. Therefore as in the sp(2) case, the abelian U(1) gauge field can be interpreted as a superposition of abelian hermitean-Einstein instantons for which F is in su(3) along hyper-planes in $\mathbb{E}^7$. This interpretation is consistent with the power law decay of the gauge field at infinity.

The U(1) gauge fields (3.9) can be used to rotate either intersecting brane configurations or appropriately compactified fundamental strings and M-2-branes. The rotation of the spacetime associated with the spherically symmetric solution

$$A_i = v^k \varphi_k{}^j{}_i \partial_j \frac{\mu}{|x_v|_v^4}\,,$$

[3] If f is allowed to depend on x^7, then the equation for f is not that of a harmonic function and so the corresponding solutions are not of the desirable type.

is

$$(\mathcal{J})_{ij} = \mu v^k \varphi_{kij}\,,$$

where we have performed the integration at the S^5 sphere at infinity of the hyper-plane P_v.

3.4 $h = \mathrm{so}(8)$

There are many solutions to the Maxwell equations that include for example electro-magnetic waves. Here we shall seek solutions for which F spans other subalgebras of so(8) from those that we have mentioned in the previous sections. Such configurations as solutions of the Maxwell theory are not supersymmetric. Moreover, we shall require that they resemble the supersymmetric solutions that we have constructed in the previous sections. An example of this is to consider a slight generalization of the hermitean-Einstein condition on F by letting F to take values in U(n) instead of the su(n) subalgebra of so($2n$), i.e.

$$F_{ab} = \Lambda(\omega_K)_{ab}\,. \tag{3.10}$$

The Maxwell field equations imply that Λ is constant. Again F is a (1,1)-form with respect to the complex structure K and a straightforward computation reveals that a solution of (3.10) is

$$A_a = K^b{}_a \partial_b f - \frac{1}{2}\Lambda(\omega_K)_{ab} x^b\,, \tag{3.11}$$

where f is that of (3.5). The advantage of considering (3.10) is that it admits an obvious generalization on hermitean manifolds which include the Calabi-Yau and hyper-Kähler manifolds. The disadvantage of such solution is that the second term in (3.11) may lead to spacetimes that are not asymptotically flat.

An alternative way to construct non-supersymmetric solutions is by superposing the supersymmetric solutions found in the previous sections using the linearity of Maxwell equations. To demonstrate this, we shall first construct a solution for which F takes values in so(4) by superposing and instanton and anti-instanton solution. For this we introduce two commuting quaternionic structures $\{I_r; r = 1,2,3\}$ and $\{J_r; r = 1,2,3\}$ on $\mathbb{R}^4$ associated with the decomposition so(4) = su(2)$\oplus$su(2). Then a solution with F in so(4) is

$$A_a = \sum_{i=1}^{N}\sum_{r=1}^{3} (I_r)^b{}_a \partial_b \frac{\mu_i^r}{|x-x_i|^2} + \sum_{i=1}^{N'}\sum_{r=1}^{3} (J_r)^b{}_a \partial_b \frac{\tilde{\mu}_i^r}{|x-x_i|^2}\,.$$

These abelian U(1) gauge field can be used to rotate NS-5-branes[4] and compactified strings which will lead to rotating black holes in five-dimensions. A spherically symmetric U(1) gauge field is

$$A_a = \sum_{r=1}^{3} [\mu^r (I_r)^b{}_a + \tilde{\mu}^r (J_r)^b{}_a] \partial_b \frac{1}{|x|^2}$$

[4]To find such so(4) rotating NS-5-brane solutions the ansatz (2.2) that we are using has to be modified. Such a modification may lead to a U-duality invariant ansatz.

and the angular momentum of the associated spacetime is

$$(\mathcal{J})_{ab} = \sum_{r=1}^{3} \left[\mu^r (I_r)_{ab} + \tilde{\mu}^r (J_r)_{ab}\right].$$

Observe that the angular momentum is a vector in so(4) and it has two independent eigenvalues equal to the rank of so(4).

The above procedure can be easily generalize to superpose hermitean-Einstein connections A to construct solutions for which F is in so(8). For this we consider complex structures $\{K_q; q = 1, \dots, \ell\}$ on $\mathbb{E}^8$ and superpose the hermitean-Einstein connections with respect to each K_q. A solution is

$$A_a = \sum_{q=1}^{\ell} (K_q)^b{}_a \partial_b f_q \tag{3.12}$$

where $\{f_q; q = 1, \dots, \ell\}$ are distinct harmonic functions similar to those in (3.5) for $n = 4$. Then F spans the subspace $\cup_{q=1}^{\ell}$su(4) $\subset$ so(8), where each su(4) is associated with a complex structure $\{K_q; q = 1, \dots, \ell\}$. There is always a choice of complex structures $\{K_q; q = 1, \dots, \ell\}$ such that F spans so(8). The U(1) gauge field (3.12) can be used to rotate a fundamental string or a M-2-brane. A spherically symmetric solution is

$$A_a = \sum_{q=1}^{\ell} (K_q)^b{}_a \partial_b \frac{\mu_q}{|x|^6}$$

and the angular momentum of the associated spacetime is

$$(\mathcal{J})_{ab} = \sum_{q=1}^{\ell} \mu_q (K_q)_{ab}\,.$$

In general, $\mathcal{J}$ would have four independent eigenvalues equal to the rank of so(8).

We can also do similar superpositions using connections for which F is in sp(2). For this consider for example the two quaternionic structures on $\mathbb{E}^8$,

$$\begin{aligned} \mathbf{I}_r{}^{i\mu}{}_{j\nu} &= \delta^i{}_j (I_r)^\mu{}_\nu \\ \mathbf{J}_r{}^{i\mu}{}_{j\nu} &= \delta^i{}_j (J_r)^\mu{}_\nu\,, \end{aligned} \tag{3.13}$$

where $\{I_r; r = 1, 2, 3\}$ and $\{J_r; r = 1, 2, 3\}$ are those of the so(4) case above. Then the curvature F of the connection

$$A_{i\mu} = \sum_{\{p_i, a\}} \sum_{r=1}^{3} (I_r)^\nu{}_\mu \partial_{i\nu} \frac{\mu^r(p,a)}{|p_i x^{i\mu} - a^\mu|^2} + \sum_{\{\tilde{p}_i, \tilde{a}\}} \sum_{r=1}^{3} (J_r)^\nu{}_\mu \partial_{i\nu} \frac{\tilde{\mu}^r(\tilde{p},\tilde{a})}{|\tilde{p}_i x^{i\mu} - \tilde{a}^\mu|^2}\,,$$

spans the subspace sp(2) $\cup$ sp(2) $\subset$ so(8), where each sp(2) is associated with the quaternionic structures (3.13) above. It is clear that there are many other ways of constructing new solutions for example by superposing the different methods proposed above. The angular momentum induced on branes or intersecting branes by these U(1) gauge fields can be easily computed and we shall not present the result here. The above results can be easily extended to any suitable dimension.

3.5 Strings, waves and rotations

It is straightforward to superpose the rotated spacetimes we have constructed with a pp-wave and a fundamental string for $\mathcal{M}_{(8)} = \mathbb{E}^8$. The field equations simply imply that the functions g_1 and g_2 associated with the string and the wave are harmonic functions on $\mathbb{E}^8$. Moreover, the field equations for A do not alter with the addition of the string and the wave and so the expressions that we have given for A in the previous sections are still valid. The supersymmetry preserved by such solutions depends on the type of rotation and whether the configuration involves a wave or a string or both. Since the supersymmetry of these configurations without rotation has been investigated in the literature, here we shall consider three cases of solutions with (i) rotation and wave, (ii) rotation and string and (iii) rotation, wave and string. We shall do the analysis of the equations without reference to a particular eight-dimensional manifold and only when we give the solutions we shall specialize to $\mathcal{M}_{(8)} = \mathbb{E}^8$. We shall also choose $\lambda = -1/2$.

(i) For such configurations, we allow F, g_2 to depend on $x \in \mathcal{M}_{(8)}$ and set $g_1 = 1$. The third Killing spinor equation in (2.3) implies that

$$\Gamma^{\underline{v}} \epsilon_\pm = 0 \tag{3.14}$$

and the remaining reduce to

$$\begin{aligned} F_{ch} \tilde{e}^c{}_{\underline{a}} \tilde{e}^h{}_{\underline{b}} \Gamma^{\underline{ab}} \epsilon_- &= 0 \\ D^{(\pm)}_{\underline{a}} \epsilon_\pm &= 0 \\ \left(\Gamma^{\underline{a}} \tilde{e}^a{}_{\underline{a}} \partial_a \phi_{(8)} \mp \frac{1}{3!} h_{abc} \tilde{e}^a{}_{\underline{a}} \tilde{e}^b{}_{\underline{b}} \tilde{e}^c{}_{\underline{c}} \Gamma^{\underline{abc}} \right) \epsilon_\pm &= 0\,. \end{aligned}$$

So for $\mathcal{M}_{(8)} = \mathbb{E}^8$, these imply that $\epsilon_\pm$ are constant which satisfy (3.14) and in addition ϵ_- satisfies the second equation in (3.1), i.e. $F_{ab}\Gamma^{ab}\epsilon_- = 0$. Using the decomposition (3.3), the solutions of condition (3.14) lie either in the subspace $(\mathbf{1}_s, \mathbf{8}_s) \oplus (\mathbf{1}_s, \mathbf{8}_c)$ or in the subspace $(\mathbf{1}_c, \mathbf{8}_c) \oplus (\mathbf{1}_c, \mathbf{8}_s)$ of $\mathbf{16}_s \oplus \mathbf{16}_c$. The fraction of supersymmetry preserved depends on the way we embed the subalgebra h that F lies in so(8). If we use the embeddings of [27], then the fractions of supersymmetry preserved for spinors in the subspaces $(\mathbf{1}_s, \mathbf{8}_s) \oplus (\mathbf{1}_s, \mathbf{8}_c)$ and $(\mathbf{1}_c, \mathbf{8}_c) \oplus (\mathbf{1}_c, \mathbf{8}_s)$ are summarized in the table below, respectively.

Algebra	so(8)	spin(7)	g_2	sp(2)	su(4)	su(3)	su(2)
Susy	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{10}{32}$	$\frac{3}{8}$
Susy	$\frac{1}{4}$	$\frac{9}{32}$	$\frac{10}{32}$	$\frac{11}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{3}{8}$

(ii) For such configurations, we allow F, g_1 to depend on $x \in \mathcal{M}_{(8)}$ and set $g_2 = 0$. The first and fourth Killing spinor equations in (2.3) imply that

$$\Gamma^{\underline{v}}\epsilon_+ = 0\,, \qquad \Gamma^{\underline{u}}\epsilon_- = 0\,, \tag{3.15}$$

respectively. To continue, we set[5]

$$\epsilon_+ = g_1^{-\frac{1}{2}}\eta_+\,, \qquad \epsilon_- = \eta_- \tag{3.16}$$

Then, using $\Gamma^{\underline{uv}} + \Gamma^{\underline{vu}} = 2$, the rest of the Killing spinor equations in (2.3) reduce to

$$\begin{aligned} g_1^{-1} F_{ch}\tilde{e}^c{}_{\underline{a}}\tilde{e}^h{}_{\underline{b}}\Gamma^{\underline{ab}}\eta_- &= 0 \\ D_a^{(\pm)}\eta_\pm &= 0 \\ (\Gamma^{\underline{a}}\tilde{e}^a{}_{\underline{a}}\partial_a\phi_{(8)} \mp \frac{1}{3!}h_{abc}\tilde{e}^a{}_{\underline{a}}\tilde{e}^b{}_{\underline{b}}\tilde{e}^c{}_{\underline{c}}\Gamma^{\underline{abc}})\eta_\pm &= 0\,. \end{aligned}$$

So for $\mathcal{M}_{(8)} = \mathbb{E}^8$, these imply that $\eta_\pm$ are constant which satisfy (3.15) and in addition ϵ_- satisfies $F_{ab}\Gamma^{ab}\epsilon_- = 0$. Using the decomposition (3.3), the solutions of the condition (3.15) lie either in the subspace $(\mathbf{1}_s, \mathbf{8}_s) \oplus (\mathbf{1}_c, \mathbf{8}_s)$ or in the subspace $(\mathbf{1}_c, \mathbf{8}_c) \oplus (\mathbf{1}_s, \mathbf{8}_c)$ of $\mathbf{16}_s \oplus \mathbf{16}_c$. Choosing the embedding of h in so(8) as in case (i), the fractions of supersymmetry preserved in each of the above two cases are summarized in the table below, respectively.

Algebra	so(8)	spin(7)	g$_2$	sp(2)	su(4)	su(3)	su(2)
Susy	$\frac{1}{4}$	$\frac{9}{32}$	$\frac{10}{32}$	$\frac{11}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{3}{8}$
Susy	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{10}{32}$	$\frac{3}{8}$

(iii) For solutions with rotation, fundamental string and wave, the third and fourth Killing spinor equations of (2.3) imply that

$$\Gamma^{\underline{v}}\epsilon_+ = 0\,, \qquad \epsilon_- = 0\,. \tag{3.17}$$

the rest of the Killing spinor equations reduce to

$$\begin{aligned} D_a^{(+)}\eta_+ &= 0 \\ \left(\Gamma^{\underline{a}}\tilde{e}^a{}_{\underline{a}}\partial_a\phi_{(8)} - \frac{1}{3!}h_{abc}\tilde{e}^a{}_{\underline{a}}\tilde{e}^b{}_{\underline{b}}\tilde{e}^c{}_{\underline{c}}\Gamma^{\underline{abc}}\right)\eta_+ &= 0\,, \end{aligned}$$

where we have set $\epsilon_+ = g_1^{-\frac{1}{2}}\eta_+$. So for $\mathcal{M}_{(8)} = \mathbb{E}^8$, these imply that η_+ is constant which satisfies (3.17). Such solution preserves 1/4 of supersymmetry for any choice of solution for A.

[5] The expression is asymmetric in ϵ_+ and ϵ_- which is not conventional. However it is due to the choice of the frame for our metric. One can find a symmetric expression using a frame rotation.

4. Rotations in toric hyper-Kähler spaces

Toric hyper-Kähler eight-dimensional spaces admit two tri-holomorphic commuting Killing vector fields. To describe the metric of such space, we introduce the coordinates $\{(\tau_i, y^{ir}); i = 1,2; r = 1,2,3\}$ on $\mathcal{M}_{(8)}$, where τ_i are the coordinates adopted along the Killing vector fields. Then the metric [28, 12] is

$$ds^2_{(8)} = U^{ij}(d\tau_i + \omega_i)(d\tau_j + \omega_j) + U_{ij}\delta_{rs}dy^{ir}dy^{js}\,, \tag{4.1}$$

where $\omega_i = \omega_{i,jr}dy^{jr}$, $U^{ij}U_{jk} = \delta^i{}_k$ and

$$\begin{aligned}\partial_{ir}U_{jk} &= \partial_{jr}U_{ik}\\ \delta^{tu}\epsilon_{trs}\partial_{iu}U_{jk} &= \partial_{[ir}\omega_{|k|,jr]}\,.\end{aligned}$$

One can find that

$$U_{ij} = U^{\infty}_{ij} + \sum_{\{(p,a)\}}\mu((p,a))\frac{p_ip_j}{|p_iy^i - a|}\,,$$

where $\{U^{\infty}_{ij}\}$ is suitably chosen constant matrix and $\{(p_i, a^r); i = 1,2; r = 1,2,3\}$ are the parameters of the solution. The interpretation of this solution in the context of strings or M-theory is as superposition of KK-monopoles at Sp(2) angles. We can easily add a wave and a string in the configurations in which case g_1 and g_2 will be harmonic functions with respect to the (4.1). This equation is considerably simplified if we take g_1 and g_2 to be invariant under the Killing isometries. In this case, g_1 satisfies

$$U^{ij}\delta^{rs}\partial_{ir}\partial_{js}g_1 = 0$$

and similarly for g_2. A solution of this equation is

$$g_1 = 1 + \sum_{\{(p,a)\}}\mu((p,a))\frac{1}{|p_iy^i - a|}\,,$$

and similarly for g_2. The parameters $\{(p,a)\}$ of the metric, g_1 and g_2 may be different. But if we require for a wave and for a string to lie on each KK-monopole involved in the superposition, the we must take all of them to be the same up to a scale.

It remains to solve the equation for the rotation. In this case there are two natural Maxwell field strengths with values in sp(2). The gauge potentials are the duals of the Killing vector fields of the toric hyper-Kähler metric, i.e.

$$A^{(i)} == U^{ij}(d\tau_j + \omega_j)\,.$$

The associated two-form field strengths $F^{(i)} = dA^{(i)}$ are in sp(2) because the Killing vectors fields are tri-holomorphic. Moreover $F^{(i)}$, $i = 1,2$, solve the Maxwell field equations either as consequence of the Jacobi identity together with the condition that they lie in sp(2) or equivalently because the toric hyper-Kähler spaces are Ricci flat [29]. The most general A that we can substitute in our ansatz is a linear combination

$$A = c_iA^{(i)} = c_iU^{ij}(d\tau_j + \omega_j)$$

of the two U(1) gauge potentials above, where $\{c_i, i = 1,2\}$ is a constant vector.

5. Rotations and intersecting branes

There are three different orthogonal intersections of NS-5-branes. These are (i) any two NS-5-branes intersecting on a 3-brane, (ii) any three NS-5-branes intersecting on a string and (iii) any two NS-5-branes intersecting at a string. The latter intersection can be generalized to a multiple NS-5-brane intersection at Sp(2) angles. All these intersections are associated with a $\mathcal{M}_{(8)}$ geometry. In what follows, we shall be mainly concerned with Sp(2)-intersecting NS-5-branes on a string. To describe the eight-dimensional geometry that arise in this case, we introduce coordinates $\{x^{i\mu}; i = 1,2; \mu = 1,\dots,4\}$ on $\mathcal{M}_{(8)}$ and two quaternionic structures

$$\begin{aligned} \mathbf{J}_r{}^{i\mu}{}_{j\nu} &= \delta^i{}_j J_r{}^\mu{}_\nu \\ \mathbf{I}_r{}^{i\mu}{}_{j\nu} &= \delta^i{}_j I_r{}^\mu{}_\nu \,, \end{aligned}$$

where $\{J_r; r = 1,2,3\}$ and $\{I_r; r = 1,2,3\}$ are the quaternionic structures on $\mathbb{R}^4$ associated with the anti-self-dual and self-dual two-forms, respectively. We remark that

$$[J_r, I_s] = 0\,. \tag{5.1}$$

We can arranged without loss of generality that for this intersection, $\mathcal{M}_{(8)}$ is a hyper-Kähler manifold with torsion (HKT) with respect to pair $(\nabla^{(+)}, \mathbf{J}_r)$. This implies that the connection $\nabla^{(+)}$ has holonomy Sp(2). The metric on $\mathcal{M}_{(8)}$ can be written as

$$ds^2_{(8)} \equiv \gamma_{ab} dx^a dx^b = (U_{ij}\delta_{\mu\nu} + V^r{}_{ij}(I_r)_{\mu\nu}) dx^{i\mu} dx^{j\nu}\,, \tag{5.2}$$

where the matrices U, V^r have been given in [13] and $i,j = 1,2;\ \mu,\nu = 1,\dots,4$. Due to (5.1), the above metric is hermitean with respect to $\mathbf{J}_r$ as required. These geometries are constructed by pulling-back with maps

$$\begin{aligned} \tau :\ \mathbb{H}^2 &\longrightarrow \mathbb{H} \\ q^i\ &\rightarrow\ \tau(q^i) = p_i q^i - a \end{aligned}$$

the four-dimensional HKT geometry associated with the NS-5-brane to $\mathbb{R}^8$ and then summing up over the different choices of τ, where $q^i = x^{i1} + ix^{i2} + jx^{i3} + kx^{i4}$ and the quaternions $\{(p_i, a)\}$ are the parameters of the maps. The details of the construction of the metric $ds^2_{(8)}$ and of the 3-form $H_{(8)}$ of this geometry will not be given here. This has been done in [13] where it was also found that the dilaton is

$$e^{2\phi_{(8)}} = \gamma^{1/4}\,. \tag{5.3}$$

The position of the NS-5-branes involved in the superposition are given by the kernels of the above maps τ. For later use we remark that the metric (5.2) above has the property

$$\gamma^{bc}\partial_a\gamma_{bc} = 4\gamma^{bc}\partial_b\gamma_{ac}\,. \tag{5.4}$$

To add a wave and a string to the above brane intersection, we have to solve the equations for g_1 and g_2 in (2.4). Using (5.4) and the expression for the dilaton (5.3), we find that these equations reduce to

$$\begin{aligned}\gamma^{i\mu,j\nu}\partial_{i\mu}\partial_{i\nu}g_1 &= 0\\ \gamma^{i\mu,j\nu}\partial_{i\mu}\partial_{i\nu}g_2 &= 0\,.\end{aligned} \tag{5.5}$$

A class of solutions for these equations is

$$g_1 = 1 + \sum_{\{(p,a)\}} \frac{\mu((p,a))}{|p_i q^i - a|^2}$$

and similarly for g_2. In order for a wave and a string to lie on one NS-5-branes, the parameters $\{(p,a)\}$ of g_1 and g_2 should be chosen to be the same as those of the maps τ used to construct the metric (5.2) above. However, for the most general solution of this type the parameters $\{(p,a)\}$ of g_1, g_2 and the metric can be different.

It remains to add rotation to the configuration by solving (2.4) for F. Since $\mathcal{M}_{(8)}$ is a complex manifold with respect to three different complex structures $\{\mathbf{J}_r\}$, we shall seek solutions for which A is a hermitean-Einstein connection with respect to a complex structure $\mathbf{K}$. An obvious choice for $\mathbf{K}$ is as a linear combination of $\{\mathbf{J}_r\}$. However, unlike the situation in the flat case, here the hermitean-Einstein condition on F with respect to such a choice of $\mathbf{K}$ does *not* imply the field equations (2.4). The difficulty lies in the presence of torsion. To proceed, we shall consider special cases of the geometry (5.2) above for which the manifold $\mathcal{M}_{(8)}$ admits at least another Kähler structure with torsion (KT) with respect to a pair $(\nabla^{(-)}, \mathbf{I})$. We can then show by direct computation using the Jacobi Identities, the equation (5.4) and the relation between the metric and torsion in a KT geometry that if

$$F_{ab} = \Lambda(\omega_{\mathbf{I}})_{ab}\,,$$

then F satisfies the field equations (2.4), where Λ is a constant.

There are two special cases that we shall consider:

(i) We shall require that the geometry $\mathcal{M}_{(8)}$ admits another KT structure with respect to the pair[6] $(\nabla^{(-)}, \mathbf{I}_3)$. If this is the case, then the holonomy of $\nabla^{(-)}$ is SU(4). To construct the metric and torsion of this geometry, we simply require that the parameters $\{p_i; i = 1,2\}$ of the linear maps τ are complex numbers instead of quaternions. The metric (5.2) becomes

$$ds^2_{(8)} \equiv \gamma_{ab}dx^a dx^b = (U_{ij}\delta_{\mu\nu} + V_{ij}(I_3)_{\mu\nu})dx^{i\mu}dx^{j\nu}\,,$$

[6] We can obviously instead of $\mathbf{I}_3$ choose any of the other two complex structures $\mathbf{I}_1$ and $\mathbf{I}_2$ or even a linear combination of all three complex structures. But all these cases are symmetric, so by choosing $\mathbf{I}_3$ there is no loss of generality.

where $V = V_3$ and $V_1 = V_2 = 0$. Choosing complex coordinates with respect to $\mathbf{I}_3$, the metric (5) is rewritten as

$$ds^2_{(8)} \equiv \gamma_{ab} dx^a dx^b = Z_{ij} \delta_{\alpha\bar{\beta}} dz^{i\alpha} dx^{j\bar{\beta}} = (U_{ij} + iV_{ij}) \delta_{\alpha\bar{\beta}} dz^{i\alpha} dx^{j\bar{\beta}} .$$

The hermitean-Einstein condition for $\mathbf{K} = \mathbf{I}_3$ implies that

$$\begin{aligned} F_{i\alpha, j\beta} &= 0 \\ Z^{ij} \delta^{\alpha\bar{\beta}} F_{i\alpha, j\bar{\beta}} &= 0 \,. \end{aligned} \tag{5.6}$$

We again solve the first equation by setting

$$\begin{aligned} A_{i\alpha} &= \partial_{i\alpha} U \\ A_{i\bar{\alpha}} &= \partial_{i\bar{\alpha}} \bar{U} \end{aligned}$$

Substituting this in the second equation in (5.6), we find that

$$Z^{ij} \delta^{\alpha\bar{\beta}} \partial_{i\alpha} \partial_{j\bar{\beta}} (U - \bar{U}) = 0$$

or

$$\gamma^{i\mu, j\nu} \partial_{i\mu} \partial_{j\nu} (U - \bar{U}) = 0 \tag{5.7}$$

in real coordinates. This is precisely the equation that g_1 and g_2 for the string and the wave satisfy, so it can be solved in a similar way. To summarize, the solution is

$$\begin{aligned} ds^2 &= 2 g_1^{-1} dv(-du + A + g_2 dv) + (U_{ij} \delta_{\mu\nu} + V_{ij} (I_3)_{\mu\nu}) dx^{i\mu} dx^{j\nu} \\ H &= -\frac{1}{2} dv \wedge d(g_1^{-1} A) - \frac{1}{2} du \wedge dv \wedge d g_1^{-1} + H_{(8)} \\ e^{2\phi} &= g_1^{-1} \Big[\det(U_{ij} \delta_{\mu\nu} + V_{ij} (I_3)_{\mu\nu}) \Big]^{1/4} , \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} g_1 &= 1 + \sum_{\{(p,a)\}} \frac{\mu_1((p,a))}{|p_i q^i - a|^2} \\ g_2 &= \sum_{\{(p,a)\}} \frac{\mu_2((p,a))}{|p_i q^i - a|^2} \\ A_{i\mu} &= (I_3)^\nu{}_\mu \partial_{i\nu} \left[\sum_{\{(p,a)\}} \frac{\mu((p,a))}{|p_i q^i - a|^2} \right] , \end{aligned} \tag{5.9}$$

and $\{p_i\}$ are complex numbers. We remark that for the most general solution of this type the parameters $\{(p,a)\}$ of g_1, g_2 and of the metric can be different. The physical interpretation of the solution above is that for every pair of parameters (p, a) it corresponds a wave with charge $\mu_2((p,a))$ on string with charge $\mu_1((p,a))$ both on a 5-brane with charge $\mu_5((p,a))$ located at

$$p_i q^i - a = 0\,.$$

In addition each 5-brane has angular momentum

$$\mathcal{J} = \mu((p,a)) I_3\,.$$

Notice that for every pair of 5-branes associated with the maps (τ, τ') intersect on a string located at $\tau^{-1}(0) \cap \tau'^{-1}(0)$. So the solutions has the interpretation of waves on strings on rotating Sp(2)-intersecting NS-5-branes.

The fraction of spacetime supersymmetry preserved by solutions (5.9) is summarized in the following table:

Rotation	$\surd$	$-$	$-$	$\surd$	$\surd$	$-$	$\surd$
String	$-$	$\surd$	$-$	$\surd$	$-$	$\surd$	$\surd$
wave	$-$	$-$	$\surd$	$-$	$\surd$	$\surd$	$\surd$
Susy	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{3}{32}$	$\frac{5}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{3}{32}$
Susy	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	0

The two rows with the fractions of supersymmetry correspond to the two possibilities of letting the solutions of $\Gamma^{\underline{v}}\epsilon_+ = 0$ to be either in $(\mathbf{1}_s, \mathbf{8}_s)$ or in $(\mathbf{1}_c, \mathbf{8}_c)$ subspaces of $\mathbf{16}_s$, respectively. For computing the above fractions of supersymmetry, we have used the orientation on $\mathbb{E}^8$ that it is induced from the complex structures $\mathbf{J}_r$ to write the chirality operator in eight-dimensions. Moreover observe that the complex structures $\mathbf{I}_r$ give the same orientation on $\mathbb{E}^8$ as that of the complex structures $\mathbf{J}_r$. This is unlike the situation in $\mathbb{E}^4$ where the self-dual I_r and anti-self-dual J_r complex structures induce opposite orientations.

(ii) We shall require that the geometry $\mathcal{M}_{(8)}$ admits another HKT structure with respect to $(\nabla^{(-)}, \mathbf{I}_r)$. In this case the holonomy of both connections $\nabla^{(-)}$ and $\nabla^{(-)}$ is Sp(2). To construct the metric and torsion of this geometry, we simply require that the parameters $\{p_i; i = 1, 2\}$ of the linear maps τ are real numbers instead of quaternions [12, 13]. The metric (5.2) becomes

$$ds^2_{(8)} \equiv \gamma_{ab} dx^a dx^b = U_{ij} \delta_{\mu\nu} dx^{i\mu} dx^{j\nu}\,,$$

where $V_1 = V_2 = V_3 = 0$. Now we take F to be in Sp(2), i.e.

$$F_{i\mu, j\nu} (I_r)^{\mu}{}_{\rho} (I_r)^{\nu}{}_{\sigma} = F_{i\rho, j\sigma} \tag{5.10}$$

(no summation over r). Since this is a special case of the hermitean-Einstein condition, any connection that satisfies equation (5.10) also solves the field equations (2.4). The equation (5.10) can be solved as in the flat case. So to summarize, the solution in this case is

$$\begin{aligned} ds^2 &= 2g_1^{-1}dv(-du + A + g_2 dv) + U_{ij}\delta_{\mu\nu}dx^{i\mu}dx^{j\nu} \\ H &= -\frac{1}{2}dv\wedge d(g_1^{-1}A) - \frac{1}{2}du\wedge dv\wedge dg_1^{-1} + H_{(8)} \\ e^{2\phi} &= g_1^{-1}\det(U_{ij})\,, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} g_1 &= 1 + \sum_{\{(p,a)\}} \frac{\mu_1((p,a))}{|p_i q^i - a|^2} \\ g_2 &= \sum_{\{(p,a)\}} \frac{\mu_2((p,a))}{|p_i q^i - a|^2} \\ A_{i\mu} &= \sum_{\{(p,a)\}} \sum_{r=1}^{3} (I_r)^\nu{}_\mu \partial_{i\nu} \frac{\mu^r((p,a))}{|p_i q^i - a|^2}\,, \end{aligned} \tag{5.12}$$

and $\{p_i; 1,2\}$ are real numbers. The physical interpretation of this solution is similar to that we have given for the solution (5.8) of the previous case. The angular momentum of each 5-brane involved in the intersection is

$$\mathcal{J} = \sum_{r=1}^{3} \mu^r((p,a))I_r\,.$$

The supersymmetry preserved by the solutions (5.11 is summarized in the following table:

Rotation	$\surd$	$-$	$-$	$\surd$	$\surd$	$-$	$\surd$
String	$-$	$\surd$	$-$	$\surd$	$-$	$\surd$	$\surd$
wave	$-$	$-$	$\surd$	$-$	$\surd$	$\surd$	$\surd$
Susy	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{32}$	$\frac{3}{16}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{3}{32}$
Susy	$\frac{3}{32}$	0	$\frac{3}{32}$	0	$\frac{3}{32}$	0	0

The two rows with the fractions of supersymmetry correspond to the two possibilities of letting the solutions of $\Gamma^{\underline{v}}\epsilon_+ = 0$ to be either in $(\mathbf{1}_s, \mathbf{8}_s)$ or in $(\mathbf{1}_c, \mathbf{8}_c)$ subspaces of $\mathbf{16}_s$, respectively. We remark that the same fractions of supersymmetry are preserved by the associated toric hyper-Kähler solutions of section 3.

A special case of this class of solutions is to take the ratios p_1/p_2 of the parameters $\{p_i; 1,2\}$ of all maps τ used to construct the background 5-brane intersecting geometry to be the same. The resulting solution has then the interpretation of parallel 5-branes. In this case, the holonomy of the connections $\nabla^{(+)}$ and $\nabla^{(-)}$ is sp(1). Moreover we can add rotations, waves and strings as above. We may also take the ratios p_1/p_2 of the parameters $\{p_i; 1,2\}$ that appear in the expressions for g_1 and g_2 in (5.12) to be the same as those of the parameters of the maps τ that are used in the construction of the metric. The resulting solution will have the interpretation of parallel 5-branes with waves and strings. The supersymmetry preserved by this solution is summarized in the following table:

Rotation	$\surd$	$-$	$-$	$\surd$	$\surd$	$-$	$\surd$
String	$-$	$\surd$	$-$	$\surd$	$-$	$\surd$	$\surd$
wave	$-$	$-$	$\surd$	$-$	$\surd$	$\surd$	$\surd$
Susy	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

The same fractions of supersymmetry are preserved for both choices of the projection $\Gamma^{\underline{v}}\epsilon_+$.

There is also a third possibility for which by suitably restricting the parameters $\{p_i\}$ of the maps τ the connection $\nabla^{(-)}$ has holonomy spin(7). This suggests that we may be able find rotations for which F is in spin(7). We shall not further explore this here.

We finally remark that the solutions (5.9) and (5.11) are not the most general ones of the Killing and field equations with the above interpretation. They should be rather thought as describing the asymptotic behaviour of more general solutions near the transverse spatial infinity. It is expected that these more general solutions exist and this will be exploited to investigate the geometry near the NS-5-brane intersections in section 8.

6. Rotations and Ricci flat cones

So far we have chosen eight-dimensional manifolds $\mathcal{M}_{(8)}$ that are asymptotically flat. However, our ansatz allows many other choices of $\mathcal{M}_{(8)}$. One of them is that of a Ricci-flat cone $C(\mathcal{N}_{(7)})$ over a seven-dimensional manifold $\mathcal{N}_{(7)}$. To preserve some supersymmetry, the manifolds $\mathcal{N}_{(7)}$ are chosen such that their associated cones have holonomy SU(n), $n = 2,3,4$ (Calabi-Yau), Sp(2) (hyper-Kähler), G_2 and spin(7). Moreover the dilaton $\phi_{(8)}$ is constant and the three-form field strength $H_{(8)}$ vanishes. The metric on $\mathcal{M}_{(8)}$ is

$$ds^2_{(8)} = dr^2 + r^2 ds^2_{(7)}\,,$$

where

$$ds^2_{(7)} = h_{ij} dy^i dy^j$$

is the metric on $\mathcal{N}_{(7)}$ and r is a radial coordinate. In many cases, $\mathcal{M}_{(8)}$ are singular at $r = 0$. Such eight-dimensional geometries have recently appear in the investigation of near horizon geometries of the M-2-brane and of the M-5-brane [15].

A class of solutions with a string and a wave can be found by assuming that g_1 and g_2 are functions of the radial coordinate r. In this case, we have that

$$\begin{aligned} g_1 &= 1 + \frac{m_1}{r^6} \\ g_2 &= \frac{m_2}{r^6} \, . \end{aligned} \tag{6.1}$$

Obviously more general solutions can be found by taking g_1 and g_2 to be general harmonic functions on $C(\mathcal{N}_{(7)})$ and therefore to depend on the coordinates of $\mathcal{N}_{(7)}$ as well. The equation of motion for F can be rewritten as

$$\begin{aligned} \partial_j \left(\sqrt{h} h^{jk} F_{kr} \right) &= 0 \\ \partial_j \left(r^3 \sqrt{h} h^{jk} h^{i\ell} F_{k\ell} \right) + \partial_r \left(r^5 \sqrt{h} h^{i\ell} F_{r\ell} \right) &= 0 \, . \end{aligned} \tag{6.2}$$

Now if $F_{ri} = 0$ and $F_{ij} = F_{ij}(y)$, then F is a harmonic two-form on $\mathcal{N}_{(7)}$. Such solutions are then associated with principal U(1) bundles $\mathcal{L}$ over $\mathcal{N}_{(7)}$. For trivial bundles, the rotation can be eliminated with a coordinate transformation.[7] So the interesting cases are those associated with non-trivial bundles. This requires that the second betti number of $\mathcal{N}_{(7)}$ to be non-vanishing. Many such examples have been constructed. These include the tri-Sasakian manifolds of [30] which give rise to hyper-Kähler cones, and their associated by squashing weak G_2 holonomy seven-dimensional manifolds which give rise to cones with spin(7) holonomy. Other examples include some of the Sasaki-Einstein spaces of [31] which are associated with Calabi-Yau cones. If A vanishes our solutions specialize to those considered in [14, 15].

It is clear that as $r \to \infty$, our solutions are asymptotically $\tilde{\mathcal{L}} \times \mathbb{R}$, where $\tilde{\mathcal{L}}$ is a principal U(1) bundle over $C(\mathcal{N}_{(7)})$ induced from the principal U(1) bundle on $\mathcal{N}_{(7)}$. Since these spacetimes are not asymptotically flat, A is not straightforwardly related to angular momentum at infinity. The fractions supersymmetry preserved by our solutions are summarized in the following table:

For the calculation of the above fractions, we have assumed that F is generic and so the Maxwell Killing spinor equation does not admit any non trivial solutions. In this case the above fractions do not alter with the addition of strings and waves in the configuration. The two supersymmetry rows correspond to the choice of taking the solutions of $\Gamma^{\underline{v}}\epsilon_+ = 0$ to be either in $(\mathbf{1}_s, \mathbf{8}_s)$ or in $(\mathbf{1}_c, \mathbf{8}_c)$, respectively.

[7] We assume that the connection is also trivial.

Holonomy	SU(2)	SU(3)	SU(4)	Sp(2)	G_2	spin(7)
Susy	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{32}$	$\frac{1}{16}$	$\frac{1}{32}$
Susy	$\frac{1}{8}$	$\frac{1}{16}$	0	0	0	0

7. M-theory

The solutions that we have described in the previous sections as solutions of IIA string theory can be easily lifted to M-theory. For this let z be the eleventh coordinate. The relevant Kaluza-Klein ansatz[8] for the reduction from eleven dimensions to ten is

$$\begin{aligned} ds^2_{(11)} &= e^{\frac{4}{3}\phi} dz^2 + e^{-\frac{2}{3}\phi} ds^2_{(10)} \\ G_4 &= H \wedge dz\,, \end{aligned} \tag{7.1}$$

where ϕ is the ten-dimensional dilaton, the ten-dimensional metric $ds^2_{(10)}$ is in the string frame, G_4 is the 4-form field strength of eleven-dimensional supergravity and H is the ten-dimensional NS$\otimes$NS three-form field strength. The lifting to M-theory of solutions given by the ansatz (2.2) is

$$\begin{aligned} ds^2_{(11)} &= g_1^{-\frac{2}{3}} \left[e^{\frac{4}{3}\phi_{(8)}} dz^2 + 2e^{-\frac{2}{3}\phi_{(8)}} dv(-du + A + g_2 dv) \right] + g_1^{\frac{1}{3}} e^{-\frac{2}{3}\phi_{(8)}} ds^2_{(8)} \\ G_4 &= \left[-\frac{1}{2} dv \wedge d(g_1^{-1} A) - \frac{1}{2} du \wedge dv \wedge dg_1^{-1} + H_{(8)} \right] \wedge dz \end{aligned}$$

for $\lambda = -1/2$. Since this solution of M-theory is constructed from lifting a solution of IIA string theory is not localized in the z direction. It turns out that we can modify our ansatz in eleven dimensions to allow localization of g_2 associated with the wave as it has been done in the special case investigated in [10]. However in what follows this property of g_2 will not be used[9] and so we shall not elaborate further on this point.

It is well known that IIA strings lift to M-theory as M-2-branes, pp-waves lift again as pp-waves and NS-5-branes lift as M-5-branes. Moreover it turns out that the rotation associated with the abelian field-strength A also lifts as a rotation. The worldvolume coordinates of the M-2-brane are (u, v, z). Following this correspondence under IIA/M-theory duality, the interpretation of the lifted solutions of the IIA solutions that we have found in the previous sections is straightforward. We remark that it is not necessary to localize A in the z coordinate. This is because the number of transverse directions of the M-2-brane is the same as number of transverse

[8] We have not included in this ansatz the Kaluza-Klein vector and the IIA four-form field strength because they vanish for our ten-dimensional solutions.

[9] For example as we shall see in section 8, the near horizon geometry of these configurations can be determined by taking g_2 to be independent of z.

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directions of the IIA string. For example the wave on a rotating string solution of section 3.5 is lifted to M-theory as

$$
\begin{aligned}
ds^2_{(11)} &= g_1^{-\frac{2}{3}}\left[dz^2 + 2dv(-du + A + g_2 dv)\right] + g_1^{\frac{1}{3}} ds^2(\mathbb{E}^8) \\
G_4 &= -\frac{1}{2}\left[dv \wedge d(g_1^{-1}A) + du \wedge dv \wedge dg_1^{-1}\right] \wedge dz\,,
\end{aligned}
$$

where g_1 and g_2 are harmonic functions on $\mathbb{E}^8$ and the field strength F of A is in su(4) or in spin(7) or in so(8). The explicit expressions of A in all three cases have been given in sections 3.1, 3.3 and 3.4, respectively. The angular momentum is the same as that we have computed in the context of IIA strings.

As another example, we lift the solutions for which $\mathcal{M}_{(8)}$ is a Ricci-flat cone of section 6. The resulting M-theory solution is

$$
\begin{aligned}
ds^2_{(11)} &= g_1^{-2/3}\left[dz^2 + 2dv(-du + A + g_2 dv)\right] + g_1^{1/3}(dr^2 + r^2 ds^2_{(7)}) \\
G_4 &= -\frac{1}{2}\left[dv \wedge d(g_1^{-1}A) + du \wedge dv \wedge dg_1^{-1}\right] \wedge dz\,,
\end{aligned}
$$

where g_1 and g_2 are given in (6.1) and A is a connection of a principal U(1) bundle on $\mathcal{N}_{(7)}$ for which F is harmonic.

Finally, the rotating rotated intersecting branes of section 5 are lifted as follows:

$$
\begin{aligned}
ds^2_{(11)} &= g_1^{-2/3}\left[e^{\frac{4}{3}\phi_{(8)}} dz^2 + 2e^{-\frac{2}{3}\phi_{(8)}} dv(-du + A + g_2 dv)\right] + \\
&\quad + g_1^{1/3} e^{-\frac{2}{3}\phi_{(8)}} (U_{ij}\delta_{\mu\nu} + V^1_{ij}(I_3)_{\mu\nu}) dx^{i\mu} dx^{j\nu} \\
G_4 &= \left[-1/2 dv \wedge d(g_1^{-1}A) - \frac{1}{2} du \wedge dv \wedge dg_1^{-1} + H_{(8)}\right] \wedge dz\,,
\end{aligned}
$$

where $e^{2\phi_{(8)}}$ is given (5.3), and g_1, g_2 and A are given in (5.9) in case (i) or in (5.12) for case (ii). For $A = 0$, the lifting has been described in [13].

8. Near horizon geometries

8.1 Sp(2)-Intersecting NS-5-branes

We shall first consider the near horizon geometry of two NS-5-branes intersecting at angles on a string associated with the maps τ_1 and τ_2, respectively. Moreover let us assume that $\tau_1^{-1}(0) \cap \tau_2^{-1}(0) = \{x\}$ is a point in $\mathbb{H}^2$; in such case the two NS-5-branes intersect on string. Next, we change coordinates in $\mathbb{H}^2$ as follows:

$$
x = \tau_1(q^i)\,, \qquad y = \tau_2(q^i)\,. \tag{8.1}
$$

This change of coordinates is an invertible transformation. The metric and three-form field strength in these new coordinates can be re-expressed as

$$
\begin{aligned}
ds^2 &= ds^2(\mathbb{E}^{(1,1)}) + ds^2_\infty + \frac{R_1^2}{|x|^2}(|dx|^2 + |x|^2 ds^2(S^3)) + \frac{R_2^2}{|y|^2}(|dy|^2 + |y|^2 ds^2(S^3)) \\
H &= R_1^2 \mathrm{Vol}(S^3) + R_2^2 \mathrm{Vol}(S^3)\,,
\end{aligned} \tag{8.2}
$$

where $\mathrm{Vol}(S^3)$ is the volume form of S^3, ds^2_∞ is the constant asymptotic metric at the transverse spatial infinity, and $R_1^2 = \mu(\tau_1)$ and $R_2^2 = \mu(\tau_2)$ are constants. Now in the limit that both $|x|^2 << R_1^2$ and $|y|^2 << R_2^2$, the constant asymptotic part of the metric can be neglected and the near horizon geometry is $\mathbb{E}^{(1,3)} \times S^3 \times S^3$. This near horizon geometry is exactly the same as that one finds for orthogonally intersecting NS-5-branes for which the asymptotic constant metric is $ds^2_\infty = |dx|^2 + |dy|^2$.

Next consider the case where more than two branes are involved in the intersection. It suffices to consider the case of three intersecting branes associated with the maps τ_1, τ_2 and τ_3, respectively. This is because the arguments that we shall present to determine the near horizon geometry in this case can be easily extended to N branes. We shall take that the sets $\tau_1^{-1}(0)$, $\tau_2^{-1}(0)$ and $\tau_3^{-1}(0)$ pairwise to intersect on a point and $\tau_1^{-1}(0) \cap \tau_2^{-1}(0) \cap \tau_3^{-1}(0) = \emptyset$, i.e. in differential topology terminology the three branes are in general position.[10] In particular, the latter condition implies that not all three branes intersect on the same string. Now as we approach the intersection region of the branes associated with say the maps τ_1 and τ_2, the metric is dominated by the x and y coordinates adapted to these two branes. This is because by assumption the intersections of all three branes are well separated. So the near horizon geometry at the intersection of the above pair of branes is $\mathbb{E}^{(1,3)} \times S^3 \times S^3$ as for the two brane intersection above, and similarly for the near horizon geometry at the intersection of the other two pairs.

8.2 Rotating Sp(2)-Intersecting NS-5- and M-5-branes

To find a smooth geometry near the pairwise Sp(2)-intersection of rotating NS-5-branes superposed with strings and waves, we have to seek more general solutions from those found in section 5. For these new solutions, the functions g_1, g_2 associated with the string and the wave, and the U(1) gauge field A are again solutions of (5.5) and (5.7), respectively, but they should have different asymptotic behaviour near the intersection regions from those of (5.9). Unfortunately, we were not able to solve the equations (5.5) and (5.7) exactly. So to achieve our purpose we shall construct an approximate solution near the horizon which has the desirable behaviour. This is related to the observation in [32] that generalized harmonic function equations simplify near horizons.

We shall first consider strings and waves in the background of two Sp(2)-intersecting NS-5-branes on a string. To proceed we shall first estimate the behaviour of the inverse of the metric γ of $\mathcal{M}_{(8)}$ associated with the solution (8.2) in the limit $|x|^2 \ll R_1^2$ and $|y|^2 \ll R_2^2$. It is easy to see that schematically

$$\gamma^{-1} = \gamma_h^{-1} + O\left(\left(\frac{r}{R}\right)^4\right),$$

[10] This assumption appears to be necessary because otherwise the metric is singular.

where γ_h^{-1} is the inverse of the metric

$$ds_h^2 = \frac{R_1^2}{|x|^2}\Big(d\bar{x}dx + |x|^2 ds^2(S^3)\Big) + \frac{R_2^2}{|y|^2}\Big(d\bar{y}dy + |y|^2 ds^2(S^3)\Big)\,,$$

i.e. the inverse of the metric near the intersection, and $(r/R)^2$ denotes ratios of the type $(|x|/R_1)^2, (|y|/R_2)^2, (|x||y|/R_2R_1)$. Using this we can estimate the solutions of g_1, g_2 and A of (5.5) and (5.7) in the same limit. In particular, a solution for g_1 is

$$g_1 = \mu_1 \frac{R_1^2 R_2^2}{r_1^2 r_2^2} + O\left(\left(\frac{R}{r}\right)^2\right) \tag{8.3}$$

and similarly for g_2, where $r_1 = |x|$ and $r_2 = |y|$. A solution for A is

$$A = \mu \frac{R_1^2 R_2^2}{r_1^2 r_2^2}(\sigma_3 + \tilde{\sigma}_3) + O\left(\left(\frac{R}{r}\right)^2\right)$$

where the Kähler form $\boldsymbol{\omega}_t$ of the complex structure $\mathbf{I}_t$ is

$$\boldsymbol{\omega}_t = \frac{1}{2}\Big[d\Big(r_1^2\sigma_t + r_2^2\tilde{\sigma}_t\Big)\Big]\,,$$

and

$$\begin{aligned} d\sigma_t &= \epsilon_{tsp}\sigma_s \wedge \sigma_p \\ d\tilde{\sigma}_t &= \epsilon_{tsp}\tilde{\sigma}_s \wedge \tilde{\sigma}_p \end{aligned}$$

are left invariant one-forms on $S^3 \times S^3$. It is clear that in both the above cases the first order term dominates in the limit $r_1^2 \ll R_1^2$ and $r_2^2 \ll R_2^2$. So the metric (5.8) becomes

$$\begin{aligned} ds^2 \sim\; & -2\mu_1 \frac{r_1^2}{R_1^2}\frac{r_2^2}{R_2^2} dv du + 2\mu\mu_1 dv(\sigma_3 + \tilde{\sigma}_3) + \\ & + 2\mu_2\mu_1 dv^2 + \frac{d^2 r_1}{r_1^2} + \frac{d^2 r_2}{r_2^2} + ds^2(S^3) + d\tilde{s}^2(S^3)\,. \end{aligned} \tag{8.4}$$

After a redefinition of the various constants, the near horizon geometry of strings and waves on two rotating Sp(2)-intersecting NS-5-branes on a string is the same as that of the associated orthogonal intersection investigated in [16, 17]. So we find after a coordinate transformation that the metric (8.4) is $AdS_3 \times S^3 \times S^3 \times \mathbb{E}$. In general the geometry near the intersection of every pair of waves on strings on N rotating Sp(2)-intersecting NS-5- branes, for which the NS-5-branes are in general position, is $AdS_3 \times S^3 \times S^3 \times \mathbb{E}$. This follows from the more detail analysis of previous section. We remark that it appears that the approximate solutions (8.3) above have the appropriate decay as $|x|^2 \gg R_1^2$ and $|y|^2 \gg R_2^2$ to match the solutions given in (5.9) at the spatial transverse infinity region. However a more detail analysis is needed to establish this.

As we have seen this solution is lifted to M-theory to another solution with the interpretation of membranes ending on rotating Sp(2)-intersecting M-5-branes superposed with waves. The investigation of the near horizon geometry in this case is very similar to the one presented above. So we shall not repeat the analysis here. In particular we find that near the intersection of every pair of M-5-branes the geometry is $AdS_3 \times S^3 \times S^3 \times E^2$. In fact it is the same as the near horizon geometry of the associated orthogonal intersection investigated in [10].

9. Null compactifications

Compactifications of M-theory (or strings) to $11-k$ dimensions involve supergravity solutions of the form $\mathbb{E}^{(1,10-k)} \times \mathcal{N}_{(k)}$, where $\mathcal{N}_{(k)}$ is a compact manifold. A large class of such solutions can be given by taking $\mathcal{N}_{(k)}$ to be one of the manifolds with special holonomy. These include manifolds with holonomy SU(n), $n = 2, 3, 4$ (Calabi-Yau), Sp(2) (hyper-Kähler), G_2 and spin(7). Such manifolds $\mathcal{N}_{(k)}$ solve the supergravity field equations and preserve some of the spacetime supersymmetry provided that we take the various form field strengths to vanish and the dilaton to be constant.

Our ansatz (2.2) allows the construction of solutions of strings and M-theory associated with new compactifications that have the topology of a U(1) principal bundle $\mathcal{L}$ over the manifolds associated with above standard compactifications. However in this case, the compact direction along the fibre direction of $\mathcal{L}$ is null and the string or membrane form field strength gets an non-zero expectation value. Null compactifications of gravity theories has been investigated in the past, see for example in [18]. Here we propose an adaptation of these compactifications in the context of strings and M-theory. To find such solutions, we set $g_1 = 1$ and $g_2 = 0$ in our ansatz (2.2). In addition, let $\mathcal{L}$ be a principal U(1) bundle over $\mathcal{N}_{(k)}$ equipped with a connection A with curvature F. Then, the manifold $\mathcal{M}_{(8)}$ has topology $\mathbb{R}^{7-k} \times \mathcal{L}$. The field equations require that F is a harmonic two-form on $\mathcal{N}_{(k)}$. Now if $\mathcal{L}$ is topologically trivial, then $F = 0$ and the solution is $\mathbb{R}^{9-k} \times S^1 \times \mathcal{N}_{(k)}$. So for non-trivial solutions we should take $\mathcal{L}$ to be topologically non-trivial. Supersymmetry imposes additional conditions on F but these depend on the properties of $\mathcal{N}_{(k)}$ and $\mathcal{L}$ and they will be investigated below.

9.1 Null toric compactifications

The simplest case to consider is $\mathcal{N}_{(k)} = T^k$. Let $\{x^a; a = 1, \ldots, k\}$ be the periodic angular coordinates on T^k with periodicities $\{r^a; a = 1, \ldots, k\}$, respectively. We write the metric of T^8 as

$$ds^2 = \gamma_{ab} dx^a dx^b \,,$$

where $\{\gamma_{ab}\}$ is a constant matrix. The field equations for A imply that $F = \frac{1}{2} F_{ab} dx^a \wedge dx^b$ is constant. This in turn implies that

$$A_a = F_{ab} x^b + c_a \,,$$

where c is a constant. Given a basis $\{C_a; a = 1, \ldots, k\}$ of $H_1(T^k, \mathbb{Z})$ adapted to this coordinate system, we find the dual basis $\{\lambda^a; a = 1, \ldots, k\}$ in $H^1(T^k, \mathbb{Z})$, where

$$\lambda^a = \frac{1}{r^a} dx^a\,;$$

(no summation over the index a). A basis in $H^2(T^k, \mathbb{Z})$ can then be found by taking

$$\lambda^{ab} = \lambda^a \wedge \lambda^b$$

for $a < b$. Rewriting F in this basis, we get

$$F = f_{ab}\lambda^{ab}\,.$$

For F to be the curvature of a line bundle it is required that all the constants f_{ab} are integers. This gives a large class of supergravity solutions with rotation. For generic choices of F such solutions preserve 1/4 of supersymmetry. There are special choices of F for which more supersymmetry is preserved by the solution. For example, one can introduce a complex structure on T^k, k even, and look for connections that satisfy the hermitean Einstein condition. There are many such connections which will lead to solutions that will preserve more supersymmetry. We shall not further elaborate on this point here.

9.2 Null Calabi-Yau compactifications

On manifolds with at least one complex structure the most natural supersymmetry condition to impose on F is to require that A is a hermitean-Einstein condition. Let $\mathcal{N}_{(k)}$ be a Calabi-Yau manifold with metric γ and complex structure I. The hermitean-Einstein condition becomes

$$\begin{aligned} F_{\alpha\beta} &= 0 \\ \gamma^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} &= 0\,, \end{aligned} \tag{9.1}$$

in complex coordinates with respect to I. Moreover, the above two conditions imply that F satisfies the Maxwell equations on $\mathcal{N}_{(k)}$. It is well known that the first condition has solutions provided that the $\mathcal{L}$ is holomorphic. So it remains to find the conditions required for the second equation in (9.1) to have solutions. For this we define the k-form

$$\lambda = \omega_I^{\frac{k}{2}-1} \wedge c_1(\mathcal{L})\,,$$

where ω_I is the Kähler form of I and $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. It turns out that a necessary and sufficient condition for the existence of a connection that satisfies the second equation in (9.1) is that

$$\int_{\mathcal{N}_{(k)}} \lambda = 0\,; \tag{9.2}$$

(see for example [33] and references within). It would be of interest to investigate the small fluctuations of string and M-theory around such compactifications. The fraction of supersymmetry preserved by such Null compactifications is the same as that of the associated Calabi-Yau ones.

These considerations can be extended to compactifications for which $\mathcal{N}_{(k)}$ is hyper-Kähler, G_2 or spin(7) [34]. It is natural to impose that F is in sp$(\frac{k}{4})$, g_2 or spin(7), respectively. It is clear that to find solutions to these equations, $\mathcal{L}$ must be restricted appropriately in a way similar to the hermitean-Einstein case in (9.2).

10. Concluding remarks

We have investigated the various ways of adding angular momentum to branes preserving supersymmetry. The resulting configurations have the interpretation of rotating branes. We have applied our methods to add rotation to Sp(2)-intersecting NS-5-branes on a string with superposed strings and pp-waves. In this way, we constructed new solutions with the interpretation of rotating rotated branes. Superpositions with strings, waves and membranes were also considered where appropriate. We then explored the related M-theory configurations. We also found approximate solutions of rotating rotated NS-5-branes in type II strings and of rotating rotated M-branes in M-theory with near horizon geometries $AdS_3 \times S^3 \times S^3 \times \mathbb{E}$ and $AdS_3 \times S^3 \times S^3 \times \mathbb{E}^2$, respectively. Our results can also be adapted in the context of D-branes by using T- and S-duality. Moreover, we have presented compact solutions that can be used for null compactifications of strings and M-theory.

Some of our solutions reduce to five-dimensional black holes. This is because in five dimensions a three-form field strength is dual to a two-form one. So, we can first reduce of our solutions to five dimensions and then dualize the three-form field strength. The resulting solutions will be a supersymmetric rotating black hole solution with various U(1) charges. It is also well known that there is a correspondence between supergravity solutions and brane worldvolume solitons. If this correspondence holds in this case, then there must be worldvolume solutions that have the interpretation rotating branes and preserve the same fraction of supersymmetry as that of the associated supergravity solutions.

A large class of supergravity solutions has been constructed by superposing or intersecting a small number of 'elementary' ones using some superposition or intersection rules. These superpositions can involve a large number of 'elementary' solutions. The success of this method can be easily demonstrated by the increasing complexity and variety of the superposed solutions. However further progress towards understanding the supergravity solutions will depend on the number and interpretation of 'elementary' solutions, as well as the development of geometric methods to treat manifolds that are equipped with natural k-forms. Global charges and their algebras may be useful to find the 'elementary' solutions [35, 36]. However no all global charges appear in the supersymmetry algebra, like for example the angular momentum for asymptotically flat spacetimes. Since most of the solutions that have been found preserve a proportion of spacetime supersymmetry, they admit connections, some with torsion, that have special holonomy. In this respect they are closely related to well known special holonomy manifolds like the hyper-Kähler and Calabi-Yau ones. If this is the case, then this correspondence indicates that many of the solutions that have been constructed represent special points in the moduli space of all possible solutions with the same holonomy and similar interpretation. Some evidence that this may be the case has been provided in [37]. However since in the context of branes the connections that appear are not the Levi-Civita ones, many of the key properties of the standard special holonomy manifolds do not generalize. Nevertheless, it is likely that new methods can be developed in the near future to investigate many of the properties of supergravity solutions.

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A. T-duality, D-branes and rotations

The type II T-duality and type IIB S-duality can be used to construct rotating D-branes beginning from the rotating fundamental IIB string of section 3.5. For this we assume that the original solution is independent from the compactifying coordinate x that the T-duality is performed and that the U(1) gauge potential A associated with the rotation does not have components along x. For example performing one S-duality and two T-dualities on the fundamental IIB string, we find the metric of a rotating D-3-brane as follows:

$$ds^2 = g_1^{-1/2}\Big(2dv(-du+A) + ds^2(\mathbb{E}^2)\Big) + g_1^{\frac{1}{2}} ds^2(\mathbb{E}^6)\,,$$

where g_1 is a harmonic function on $\mathbb{E}^6$ and A is a solution of the Maxwell field equations in $\mathbb{E}^6$. Such solutions for A have been given in section 3. In particular, one such solution is that given by requiring that F is in su(3). The D-3-brane then has angular momentum proportional to the complex structure associated with the embedding of su(3) in so(6). A preliminary computation has revealed that the near horizon geometry of this rotating D-3-brane solution may be singular. However further investigation is required to establish this.

In addition T-dualizing twice and S-dualizing once the above rotating D-3-brane, we find a new rotating NS-5-brane solution with metric

$$ds^2 = \Big(2dv(-du+A) + ds^2(\mathbb{E}^4)\Big) + g_1 ds^2(\mathbb{E}^4)\,,$$

where g_1 is harmonic function in $\mathbb{E}^4$ and A satisfies the Maxwell equations. This appears to contradict the field equations for F found in section 2. However this is not the case. The resolution of this puzzle is that there is a more general ansatz than the one we have used to add rotation to branes. This will be investigated in the future.

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